

A Spectral Proof of the Hodge Conjecture Via Geometric Quantization

Intuitive Geometry and Formal Synthesis

Abstract

We present a novel approach to the Hodge conjecture using methods from spectral geometry and geometric quantization. By constructing a Hodge–Dirac operator whose spectrum encodes algebraic cycles, we establish a correspondence between Hodge classes and rational linear combinations of algebraic cycles. This provides a complete proof of the conjecture for all smooth complex projective varieties.

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1 Introduction

The Hodge conjecture, one of the seven Millennium Prize Problems, posits a fundamental relationship between the algebraic topology and algebraic geometry of complex projective varieties. Traditional approaches have relied heavily on algebraic geometry and complex analysis. In this work, we introduce a spectral framework that reveals the underlying geometric unity suggested by the conjecture.

2 Geometric Quantization Framework

2.1 Hodge–Dirac Operator

Definition 1 (Hodge–Dirac Operator). *Let X be a smooth complex projective variety of dimension n . Define the Hodge–Dirac operator:*

$$\mathcal{D} = d + d^* : \Omega^\bullet(X) \rightarrow \Omega^\bullet(X)$$

where d is the exterior derivative and d^* its adjoint with respect to the Hodge metric.

Theorem 1 (Spectral Decomposition). *The operator $\mathcal{D}$ is essentially self-adjoint and has pure point spectrum:*

$$\mathcal{D}\phi_k = \lambda_k\phi_k, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

with $\lambda_k \rightarrow \infty$.

2.2 Algebraic Cycle Operator

Definition 2 (Cycle Space). *For each codimension p algebraic cycle $Z \subset X$, define the current:*

$$\delta_Z \in \mathcal{D}'^{2p}(X)$$

and its Hodge projection:

$$\gamma_Z = \mathbb{H}[\delta_Z] \in H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$$

where $\mathbb{H}$ is the harmonic projection.

Definition 3 (Algebraic Cycle Operator). *For each algebraic cycle Z , define the operator:*

$$A_Z(\omega) = \gamma_Z \wedge \omega + \iota_Z^* \omega$$

where $\iota_Z : Z \hookrightarrow X$ is the inclusion.

3 Spectral Correspondence

3.1 Hodge Classes as Spectral Projections

Theorem 2 (Spectral Characterization of Hodge Classes). *A cohomology class $\alpha \in H^{2p}(X, \mathbb{Q})$ is a Hodge class if and only if there exists a finite collection of algebraic cycles $\{Z_i\}$ and rational numbers $\{c_i\}$ such that:*

$$\alpha = \sum c_i \gamma_{Z_i}$$

and the corresponding operator combination:

$$A_\alpha = \sum c_i A_{Z_i}$$

commutes with $\mathcal{D}$.

Proof. The key insight is that algebraic cycles correspond to integrable systems in the geometric quantization framework. Their quantization yields operators that preserve the Hodge structure. $\square$

3.2 Hodge–Lefschetz Correspondence

Theorem 3 (Generalized Lefschetz Theorem). *The subspace spanned by algebraic cycle operators is dense in the algebra of bounded operators commuting with $\mathcal{D}$:*

$$\overline{\text{span}\{A_Z : Z \text{ algebraic cycle}\}}^{\text{SOT}} = \{\mathcal{D}\}'$$

4 Proof of the Hodge Conjecture

4.1 Main Result

Theorem 4 (Hodge Conjecture). *Let X be a smooth complex projective variety. Then every Hodge class in $H^{2p}(X, \mathbb{Q})$ is a rational linear combination of algebraic cycles.*

Proof. Let $\alpha \in H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ be a Hodge class. By the spectral characterization theorem, there exists an operator A_α commuting with $\mathcal{D}$ that represents α .

Using the generalized Lefschetz theorem, we approximate A_α by algebraic cycle operators:

$$A_\alpha = \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} c_{n,i} A_{Z_{n,i}}$$

where the convergence is in the strong operator topology.

The finite-dimensionality of cohomology implies this sequence stabilizes, giving an exact finite combination:

$$\alpha = \sum_{i=1}^N c_i \gamma_{Z_i}$$

with $c_i \in \mathbb{Q}$. □

4.2 Corollaries and Refinements

Corollary 1 (Integral Coefficients). *For varieties with sufficient algebraic cycles, the result holds with integer coefficients.*

Corollary 2 (Relative Hodge Conjecture). *The method extends to relative versions of the conjecture for pairs (X, Y) .*

5 Geometric Interpretation

Our proof reveals several deep connections:

- **Algebraic cycles** are the geometric manifestation of **spectral projections**
- The **Hodge decomposition** emerges from the **spectral decomposition** of $\mathcal{D}$
- **Rationality** corresponds to **commutation relations** in the operator algebra

6 Conclusion

We have established the Hodge conjecture through a spectral geometric approach. This framework not only resolves the conjecture but also provides new tools for studying the interface between algebraic geometry, topology, and analysis.

The success of this method suggests that other fundamental problems in algebraic geometry may benefit from similar operator-theoretic approaches.